ISRAEL JOURNAL OF MATHEMATICS 150 (2005), 357–368

DISCRETENESS CRITERIA FOR MÖBIUS GROUPS ACTING ON $\bar{R}^n$*

BY

XIANTAO WANG AND LIULAN LI

Department of Mathematics, Hunan Normal University
Changsha, Hunan 410081, People's Republic of China
e-mail: xtwang@hunnu.edu.cn, liulanli79@yahoo.com

AND

WENSHENG CAO

Institute of Mathematics, Hunan University of Science and Technology
Xiangtan, Hunan 411201, People's Republic of China
e-mail: wenscao@yahoo.com.cn

ABSTRACT

In this paper, three new discreteness criteria for Möbius groups acting on $\bar{R}^n$ are obtained; they are generalizations of known results using the information of two-generator subgroups.

1. Introduction

Let $M(\bar{R}^n)$ denote the full sense preserving Möbius group acting on $\bar{R}^n$; see [4].

In [12], Jørgensen obtained a very useful necessary condition for two-generator Kleinian groups of $M(\bar{R}^2)$, which is known as Jørgensen's inequality. As an application, he discussed the discreteness of subgroups of $M(\bar{R}^2)$ or $M(\bar{R})$ and obtained the following ([12, 13]):

* The research was partly supported by FNS of China, Program for New Century Excellent Talents in University, the Foundation for Scholars back from Foreign Countries and Scientific Research Item of Hunan Province Education Committee (No. 02C448).

Received July 5, 2004

THEOREM J_1: *A non-elementary subgroup G of $M(\bar{R}^2)$ is discrete if and only if each two-generator subgroup of G is discrete.*

THEOREM J_2: *A non-elementary subgroup G of $M(\bar{R})$ is discrete if and only if each cyclic subgroup of G is discrete.*

Furthermore, Gilman proved the following ([10]).

THEOREM G: *A non-elementary subgroup G of $M(\bar{R})$ is discrete if and only if every non-elementary subgroup generated by two hyperbolic elements of G is discrete.*

In [19], we generalized the above results as follows.

THEOREM WY_1: *A non-elementary subgroup G of $M(\bar{R}^2)$ is discrete if and only if every non-elementary subgroup generated by two loxodromic elements of G is discrete.*

Moreover, we proved ([16]):

THEOREM PW: *For a non-elementary subgroup G of $M(\bar{R}^2)$, if G contains an elliptic element of order at least 3, then G is discrete if and only if each non-elementary subgroup generated by two elliptic elements of G is discrete.*

We also proved ([20]):

THEOREM WY_2: *For a non-elementary subgroup G of $M(\bar{R}^2)$, if G contains a parabolic element, then G is discrete if and only if each non-elementary subgroup generated by two parabolic elements of G is discrete.*

See [3, 5, 8, 11] etc. for other discussions along this line.

It is interesting to generalize the above classic results into the higher dimensional case. A number of authors have addressed this question. By adding some conditions, several generalizations of theorems J_1 and J_2 in $M(\bar{R}^n)$ $(n \geq 3)$ have been obtained; see [1, 7, 9, 14, 23] etc. In [18], we generalized the known results as follows.

THEOREM WY_3: *A non-elementary subgroup G of $M(\bar{R}^n)$ is discrete if and only if $WY(G)$ (see the next section for the definition) is discrete and every non-elementary subgroup generated by two loxodromic elements in G is discrete.*

Theorem WY_3 implies that Jørgensen's convergence theorem (see [12]) is also true for subgroups of $M(\bar{R}^n)$; see [18]. That is, this result is true for general groups without the assumption of finite generation or that groups do not contain elements of finite order or that groups do not contain any convergent sequence consisting of elements with infinitely many fixed points as in the similar results of [9, 14, 25] etc.

See [6, 7, 17] etc. for other discussions along this line.

Obviously, when $n = 1$ or 2, the corresponding results in [1, 9, 14, 23] etc. do not completely coincide with the classic ones. In [9], Fang and Nai asked whether Theorem G is valid for subgroups of $M(\bar{R}^n)$ when $n \geq 3$. The examples in [17] and [18] show that, in general, Theorem G does not hold for subgroups of $M(\bar{R}^n)$ when $n \geq 3$. The main reason for the failure is that the smallest sphere S containing the limit set $L(G)$ (see [15] for the definition) of G may have codimension higher than one. So G may contain a non-discrete subgroup of rotations around S. A rotation around S is an elliptic element h in G such that h fixes S pointwise (cf. [17]).

In some sense, we can regard Theorem WY_3 as a positive answer to Fang and Nai's problem. In this paper, by using similar methods as in [6], we will prove some new discreteness criteria (Theorems 3.1, 3.2 and 3.3) which are generalizations of the known corresponding results.

2. Preliminaries

We need the following preliminaries (see [2, 18, 23] for more details).

Let Γ_n denote the n-dimensional Clifford group, $SL(2, \Gamma_n)$ the group of all n-dimensional Clifford matrices and

$$PSL(2, \Gamma_n) = SL(2, \Gamma_n)/\{\pm I\},$$

where I is the unit matrix.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \Gamma_n)$$

correspond to the mapping in $\bar{R}^n$

$$x \mapsto Ax = (ax + b)(cx + d)^{-1}.$$

This is an isomorphism between $PSL(2, \Gamma_n)$ and $M(\bar{R}^n)$. We will identify the element in $M(\bar{R}^n)$ and its corresponding element in $PSL(2, \Gamma_n)$.

For $f \in PSL(2, \Gamma_n)$, let $\tilde{f}$ denote the Poincaré extension of f (see [4]),

$$fix(f) = \{x \in \bar{R}^n : f(x) = x\}, \quad fix(\tilde{f}) = \{z \in H^{n+1} : \tilde{f}(z) = z\},$$

and let $card(M)$ denote the cardinality of set M.

Now, we give a classification to the elements of $PSL(2,\Gamma_n)$ as follows.

A nontrivial element $f \in PSL(2,\Gamma_n)$ is called

(1) **fixed-point-free** if $card[fix(f)] = 0$;

(2) **loxodromic** if $card[fix(f)] > 0$ and f can be conjugated in $PSL(2,\Gamma_n)$ to $\begin{pmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix}$, where $r > 0$, $r \neq 1$, $\lambda \in \Gamma_n$ and $|\lambda| = 1$;

(3) **parabolic** if $card[fix(f)] > 0$ and f can be conjugated in $PSL(2,\Gamma_n)$ to $\begin{pmatrix} a & b \\ 0 & a' \end{pmatrix}$, where $a, b \in \Gamma_n$, $|a| = 1$, $b \neq 0$ and $ab = ba'$;

(4) **elliptic** if $card[fix(f)] > 0$ and f can be conjugated in $PSL(2,\Gamma_n)$ to $\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}$, $u \in \Gamma_n$, $|u| = 1$ and $u \notin R$.

We call f ***g*-elliptic** if it is elliptic or fixed-point-free. Then f is g-elliptic if and only if $fix(\widetilde{f}) \neq \emptyset$.

PROPOSITION 2.1: *For a nontrivial element $f \in PSL(2,\Gamma_n)$,*

(1) *f is fixed-point-free if and only if $card[fix(\tilde{f})] = 1$; f is elliptic if and only if $card[fix(\tilde{f})] > 1$.*

(2) *$PSL(2,\Gamma_n)$ contains a fixed-point-free element if and only if n is odd and $n \geq 3$.*

A subgroup G of $PSL(2,\Gamma_n)$ is called **elementary** if it has a finite G-orbit in $\bar{H}^{n+1} = H^{n+1} \cup \bar{R}^n$ (see [4]). Otherwise, we will call G **non-elementary**.

A subgroup G of $PSL(2,\Gamma_n)$ is called **Kleinian** if it is *non-elementary and discrete*.

As a corollary of the definitions, the discussions in [24], remark B_1 and the proof of lemma B_2 in [15], we can deduce the following.

LEMMA 2.1: (1) *If G contains a loxodromic element, then G is elementary if and only if it fixes a point in $\bar{R}^n$ or a point-pair $\{x, y\} \subset \bar{R}^n$.*

(2) *If G contains a parabolic element but no loxodromic element, then G is elementary if and only if it fixes a point in $\bar{R}^n$.*

(3) *If G is purely g-elliptic, i.e., each non-trivial element of G is g-elliptic, then G fixes a point in $\bar{H}^{n+1}$.*

Remark 2.1: The case that G fixes only one point in $\bar{R}^n$ in (3) can occur when $n \geq 4$ (cf. [24]).

We obtain two corollaries.

COROLLARY 2.1: *Let $f \in PSL(2,\Gamma_n)$ be of order at least 3 or ∞ and let $g \in PSL(2,\Gamma_n)$ be conjugated to f. Then,*

(1) *if f is parabolic or loxodromic, the group $\langle f, g\rangle$ generated by f and g is elementary if and only if $\langle f, g\rangle$ fixes a point $x \in \bar{R}^n$;*

(2) *if f is fixed-point-free, $\langle f, g\rangle$ is elementary if and only if $\langle f, g\rangle$ fixes a point $x \in H^{n+1}$ or a point-pair $\{x, y\} \subset \bar{R}^n$;*

(3) *if f is elliptic, $\langle f, g\rangle$ is elementary if and only if $\langle f, g\rangle$ fixes a point $x \in \bar{H}^{n+1}$ or a point-pair $\{x, y\} \subset \bar{R}^n$.*

COROLLARY 2.2: *If $G \subset PSL(2, \Gamma_n)$ fixes no point in $\bar{R}^n$, then G is purely g-elliptic if and only if G fixes a point in H^{n+1}.*

The following lemmas will be crucial.

LEMMA 2.2 ([23]): *Let f, $g \in PSL(2, \Gamma_n)$. If $\langle f, g\rangle$ is a Kleinian group, then*

$$\|f - I\| \cdot \|g - I\| \geq 1/32.$$

LEMMA 2.3 ([6]): *Let $\{x_j\}_{j=1}^{n+1}$ be a set of $n+1$ different points in $\bar{R}^n$ and satisfy that there exists a unique $(n-1)$-sphere S in $\bar{R}^n$ containing all $n+1$ points. Let g be a Möbius transformation acting on $\bar{R}^n$ such that $g(x_j) = x_j$ for each j. Then g is either the identity or a reflection in S.*

For a non-elementary subgroup $G \subset PSL(2, \Gamma_n)$, let

$$H(G) = \{f \in G : f \text{ is loxodromic}\},$$

and if G contains some parabolic element, then let

$$P(G) = \{f \in G : f \text{ is parabolic}\}.$$

For $f \in H(G)$ or $f \in P(G)$ and $P(G) \neq \emptyset$, let

$$G_f = \{g \in G : g \text{ is conjugated to } f \text{ and} \langle f, g\rangle \text{ is non-elementary}\},$$
$$WY(G) = \{h \in G : fix(f) \subset fix(h) \text{ for all } f \in H(G)\} \quad \text{(see [18])}$$

and

$$W(G) = \{h \in G : fix(f) \subset fix(h) \text{ for all } f \in P(G)\}.$$

Then, obviously, G_f is a proper subset of $H(G)$ (or respectively $P(G)$) if $f \in H(G)$ (or respectively $f \in P(G)$) and both $W(G)$ and $WY(G)$ are purely elliptic subgroups of G.

As in [6], if G is non-elementary, we denote

$$L(I) = \{g \in G : g = I \text{ on } L(G)\},$$

where $L(G)$ is the limit set of G. Then $L(I)$ is also a purely elliptic subgroup of G.

By [4, 15, 18, 21, 22], the following are obvious.

PROPOSITION 2.2: *If G is non-elementary, then* (1) $H(G) \neq \emptyset$; (2) $W(G) = WY(G)$ *provided* $P(G) \neq \emptyset$; (3) $WY(G) = L(I)$; (4) $WY(G)$ *is discrete if and only if* $WY(G)$ *is finite.*

Remark 2.2: When $n = 1$ or 2, $WY(G) = L(I) = \{I\}$ ($= W(G)$ if $P(G) \neq \emptyset$) provided $G \subset PSL(2, \Gamma_n)$ is non-elementary; see [18].

3. Discreteness criteria for subgroups of $PSL(2, \Gamma_n)$

Now we state our main results. We will prove them in Section 5.

THEOREM 3.1: *Let $G \subset PSL(2, \Gamma_n)$ be non-elementary. Then G is discrete if and only if $WY(G)$ is discrete and each non-elementary subgroup generated by two elements of G_f is discrete, where $f \in H(G)$.*

THEOREM 3.2: *Let $G \subset PSL(2, \Gamma_n)$ be non-elementary. If $P(G) \neq \emptyset$, then G is discrete if and only if $WY(G)$ is discrete and every non-elementary subgroup generated by two elements of G_f is discrete, where $f \in P(G)$.*

Let G be a subgroup of $PSL(2, \Gamma_n)$. We say that G satisfies the **Parabolic Condition** if G contains no sequence $\{f_i\}$ such that each f_i is parabolic and $f_i \to I$ as $i \to \infty$.

THEOREM 3.3: *Let $G \subset PSL(2, \Gamma_n)$ be non-elementary and satisfy the* **Parabolic Condition**. *If G contains a g-elliptic element of order at least 3, then G is discrete if and only if $WY(G)$ is a discrete group and each non-elementary subgroup of G generated by two g-elliptic elements is discrete.*

Remark 3.1: The examples in [17, 18] show that the condition that "$WY(G)$ is a discrete group" in the above theorems cannot be removed.

Remark 3.2: When $n = 1$ or 2, by Remark 2.2, Theorem 3.1 is a generalization of Theorems G and WY_1, respectively. When $n \geq 3$, Theorem 3.1 is a generalization of Theorem WY_3.

Remark 3.3: When $n = 2$, by Remark 2.2, Theorem 3.2 is a generalization of Theorem WY_2. When $n \geq 3$, by Remark 2.2, Theorem 3.2 is a generalization of theorem 1.3 in [18]. Proposition 2.1 implies that Theorem 3.3 is a generalization of Theorem PW.

4. Several propositions

PROPOSITION 4.1: *Let $G \subset PSL(2, \Gamma_n)$ be non-elementary. If each non-elementary subgroup of G generated by two elements of G_f is discrete, where $f \in H(G)$, then G contains no sequence $\{f_i\}$ such that no f_i is elliptic and*

$$f_i \to I \quad \text{as } i \to \infty.$$

PROPOSITION 4.2: *Let $G \subset PSL(2, \Gamma_n)$ be non-elementary and $P(G) \neq \emptyset$. If each non-elementary subgroup of G generated by two elements of G_f is discrete, where $f \in P(G)$, then G contains no sequence $\{f_i\}$ such that no f_i is elliptic and*

$$f_i \to I \quad \text{as } i \to \infty.$$

PROPOSITION 4.3: *Let G be non-elementary. If each non-elementary subgroup of G generated by two g-elliptic elements is discrete and $WY(G)$ is discrete, then G contains no sequence $\{f_i\}$ with each f_i g-elliptic and*

$$f_i \to I \quad \text{as } i \to \infty.$$

Proof of Proposition 4.1: Suppose, to the contrary, that G contains such a sequence. Then for large i, we may assume that $f_i(x_0) \neq y_0$ and $f_i(y_0) \neq x_0$, where $fix(f) = \{x_0, y_0\}$.

Let

$$g_i = f_i f f_i^{-1}.$$

Then, for large enough i, $\langle f, g_i \rangle$ is elementary. Suppose this were not so, that $\langle f, g_i \rangle$ is non-elementary. Then, by our assumption, $\langle f, g_i \rangle$ is discrete. Lemma 2.2 implies that

$$||f - I|| \cdot ||g_i f^{-1} - I|| \geq 1/32.$$

This is a contradiction since $g_i f^{-1} \to I$ as $i \to \infty$.

It follows from $\langle f, g_i \rangle$ being elementary that

$$fix(f) \cap fix(f_i) \neq \emptyset.$$

Let $h_j \in G (j = 1, 2)$ be loxodromic such that

$$fix(f) \cap h_j[fix(f)] = \emptyset \quad \text{and} \quad h_1[fix(f)] \cap h_2[fix(f)] = \emptyset.$$

By replacing f by $h_j f h_j^{-1}$ in the above discussions, we know that there exists a positive number M satisfying that for $j = 1, 2$ and all $i \geq M$,

$$h_j[fix(f)] \cap fix(f_i) \neq \emptyset.$$

This shows that f_i has at least three fixed points. This is the desired contradiction.

Proof of Proposition 4.2: The proof of Proposition 4.2 follows from a discussion similar to the one in the proof of Proposition 4.1.

Proof of Proposition 4.3: Suppose, to the contrary, that G contains such a sequence. Let S be the minimal sphere containing $L(G)$, the limit set of G (see [15]). Without loss of generality, we may assume that $S = \bar{R}^k$, where $1 \leq k \leq n$.

Choose $x_j \in L(G)$ and accordingly open balls U_j in $\bar{R}^{n+1}$ $(j = 1, 2, \ldots, k+2)$ such that $x_j \in U_j$, $U_j \cap U_s = \emptyset$ whenever $j \neq s$ and for any $a_j \in U_j$, there exists only one k-sphere $S(a_1, \ldots, a_{k+2})$ containing $a_1, a_2 \ldots, a_{k+2}$.

Since $WY(G)$ is finite, we can find a g-elliptic element $h \in G$ of order at least 3 such that $[fix(h^2) \cup fix(\widetilde{h^2})] \cap U_1 = \emptyset$.

Let g_j $(j = 1, 2, \ldots, k+2)$ be loxodromic elements of G such that

(1) $fix(g_1) \subset U_1$;

(2) the attractive fixed point of g_j lies in U_j and the other fixed point lies in U_1, where $j = 2, 3, \ldots, k+2$.

Then there is an integer p such that $\widetilde{g_j}^p[fix(h^2) \cup fix(\widetilde{h^2})] \subset U_j$ $(j = 1, 2, \ldots, k+2)$.

Lemma 2.2 and our assumptions imply that for all sufficiently large i, the groups

$$\langle f_i^r, g_j^p h^r g_j^{-p} \rangle \quad (r = 1, 2)$$

are elementary.

It follows from Lemma 2.1 that for each j and each i which is large enough, $[fix(f_i) \cup fix(\widetilde{f_i})] \cap U_j \neq \emptyset$ or $[fix(f_i^2) \cup fix(\widetilde{f_i^2})] \cap U_j \neq \emptyset$. Lemma 2.3 implies that the restriction $\widetilde{f_i}|_S$ of $\widetilde{f_i}$ or $\widetilde{f_i^2}|_S$ of $\widetilde{f_i^2}$ to S is the identity. This means that f_i or $f_i^2 \in WY(G)$ since $\widetilde{f_i^r}|_S = f_i^r|_S$ $(r = 1, 2)$. This is the desired contradiction.

The proof is completed.

5. The proofs of the main results

Proof of Theorem 3.1: Suppose that G is not discrete. Then, by Proposition 4.1, G contains a sequence $\{f_i\}$ such that each f_i is elliptic and

$$f_i \to I \quad \text{as } i \to \infty.$$

Let S be the minimal sphere containing $L(G)$. Without loss of generality, we may assume that $S = \bar{R}^k$.

If $1 \leq k < n$, then we choose $x_j \in L(G)$ and accordingly open balls $U_j \subset \bar{R}^n$ $(j = 1, \ldots, k+2)$ such that $x_j \in U_j$, $U_j \cap U_r = \emptyset$ whenever $j \neq r$ and for $a_j \in U_j$ there exists only one k-sphere $S(a_1, \ldots, a_{k+2})$ containing $a_1, \ldots, a_{k+2}$.

We claim that for each j, there exists $M_j > 0$ such that for all $i > M_j$,

$$fix(f_i) \cap U_j \neq \emptyset.$$

Suppose not. Without loss of generality, we may assume that

$$fix(f_i) \cap U_j = \emptyset$$

for all i. Then we can find a loxodromic element $h \in G$ such that

$$fix(h) \subset U_j \quad \text{and} \quad h^t[fix(f)] \subset U_j$$

for all large $t > 0$.

Since

$$\|h^t f h^{-t} - I\| \cdot \|[f_i(h^t f h^{-t}) f_i^{-1}](h^t f h^{-t})^{-1} - I\| < 1/32$$

for large i, Lemma 2.2 implies that

$$\langle h^t f h^{-t}, f_i(h^t f h^{-t}) f_i^{-1} \rangle$$

is elementary. This yields

$$fix(h^t f h^{-t}) \cap fix(f_i) \neq \emptyset$$

for large enough i, i.e.,

$$h^t[fix(f)] \cap fix(f_i) \neq \emptyset.$$

This is a contradiction.

Having established the claim, we have shown that there exists $M > 0$ such that for all $i \geq M$, and each j, $fix(f_i) \cap U_j \neq \emptyset$, where $j = 1, 2, \ldots, k+2$.

Then Lemma 2.3 yields that $f_i|_S = I$, i.e., $f_i \in WY(G)$. This is the desired contradiction.

If $k = n$, then, by replacing G by $\widetilde{G}$ in the above discussions, we can also get a contradiction.

The proof is completed.

Proof of Theorem 3.2: The proof of Theorem 3.2 follows from Proposition 4.2 and a discussion similar to that in the proof of Theorem 3.1.

Proof of Theorem 3.3: Suppose that each non-elementary subgroup of G generated by two g-elliptic elements is discrete and $WY(G)$ is finite, but G itself is not discrete. Then there is a sequence $\{f_i\} \subset G$ such that

$$f_i \to I \quad \text{as } i \to \infty.$$

By Proposition 4.3, we may assume that all f_i are not g-elliptic. By passing to a subsequence, if necessary, we can obtain that $fix(f_i)$ converges in the Hausdorff metric to a one- or two-point set $Y \subset \bar{R}^n$. Then we can find a g-elliptic element $g \in G$ of order at least 3 such that

$$fix(g) \cap fix(f_i) = \emptyset$$

for each i.

Lemma 2.2 tells us that the subgroups

$$\langle g, f_i g f_i^{-1} \rangle$$

are elementary for large i since $\langle g, f_i g f_i^{-1} \rangle = \langle g, (f_i g f_i^{-1}) g^{-1} \rangle$ and $(f_i g f_i^{-1}) g^{-1} \to I$ as $i \to \infty$.

Obviously, $(f_i g f_i^{-1}) g^{-1} \neq I$ and $(f_i g^2 f_i^{-1}) g^{-2} \neq I$ for all i.

We claim that there is a subsequence of $\{(f_i g f_i^{-1}) g^{-1}\}$ or $\{(f_i g^2 f_i^{-1}) g^{-2}\}$ such that all its elements are g-elliptic.

If $card(fix(g)) = 0$, then Corollary 2.1 implies that either $fix(\widetilde{g}) \cap fix(\widetilde{f_i} \widetilde{g} \widetilde{f_i}^{-1}) \neq \emptyset$ or there exists a $\langle g, f_i g f_i^{-1} \rangle$-invariant set $S = \{x, y\} \subset \bar{R}^n$. If the first case occurs, then $fix[(\widetilde{f_i} \widetilde{g} \widetilde{f_i}^{-1}) \widetilde{g}^{-1}] \cap H^{n+1} \neq \emptyset$; if the second case occurs, then we let $g_i = (f_i g^2 f_i^{-1}) g^{-2}$, which is elliptic for each i.

If $card(fix(g)) > 0$ and $card[fix(g) \cap fix(f_i g f_i^{-1})] = 0$, then either $fix[(\widetilde{f_i} \widetilde{g} \widetilde{f_i}^{-1}) \widetilde{g}^{-1}] \cap H^{n+1} \neq \emptyset$ or $card[fix(f_i g^2 f_i^{-1}) \cap fix(g^2)] \geq 2$. That is, either $(f_i g f_i^{-1}) g^{-1}$ or $(f_i g^2 f_i^{-1}) g^{-2}$ is g-elliptic for each i.

If $card(fix(g)) > 0$ and $card[fix(g) \cap fix(f_i g f_i^{-1})] = 1$, then $\langle g, f_i g f_i^{-1} \rangle$ contains no loxodromic elements. Since $(f_i g f_i^{-1}) g^{-1} \neq I$ and $(f_i g f_i^{-1}) g^{-1} \to I$, our assumptions imply that $(f_i g f_i^{-1}) g^{-1}$ are elliptic for almost all large i.

If $card(fix(g)) > 0$ and $card[fix(g) \cap fix(f_i g f_i^{-1})] \geq 2$, then obviously $(f_i g f_i^{-1})g^{-1}$ is elliptic for each i.

Having established the claim, we have shown that G contains a sequence $\{h_i\}$ such that each h_i is g-elliptic and

$$h_i \to I \quad \text{as } i \to \infty.$$

Proposition 4.3 implies that this is the desired contradiction.

ACKNOWLEDGEMENT: The authors heartily thank the referee for a careful reading of the paper as well as for some useful comments and suggestions.

References

[1] W. Abikoff and A. Haas, *Nondiscrete groups of hyperbolic motions,* The Bulletin of the London Mathematical Society **22** (1990), 233–238.

[2] L. V. Ahlfors, *On the fixed points of Möbius transformations in $\bar{R}^n$*, Annales Academiae Scientiarum Fennicae. Series A I. Mathematica Dissertationes **10** (1985), 15–27.

[3] L. Baribeau and T. Ransford, *On the set of discrete two-generator groups,* Mathematical Proceedings of the Cambridge Philosophical Society **128** (2000), 245–255.

[4] A. F. Beardon, *The Geometry of Discrete Groups,* Graduate Text in Mathematics, Vol. 91, Springer, Berlin, 1983.

[5] A. F. Beardon, *Some remarks on nondiscrete Mobius groups,* Annales Academiae Scientiarum Fennicae. Series A I. Mathematica Dissertationes **21** (1996), 69–79.

[6] M. Chen, *Discreteness and convergence of Möbius groups,* Geometriae Dedicata **104** (2004), 61–69.

[7] S. S. Chen and L. Greenberg, *Hyperbolic spaces,* in *Contributions to Analysis* (L. Ahlfors et al., eds.), Academic Press, New York and London, 1974, pp. 49–87.

[8] Y. Chu and X. Wang, *The discrete and nondiscrete subgroups of $SL(2,R)$ and $SL(2,C)$,* Hokkaido Mathematical Journal **30** (2001), 649–659.

[9] A. Fang and B. Nai, *On the discreteness and convergence in n-dimensional Möbius groups,* Journal of the London Mathematical Society **61** (2000), 761–773.

[10] J. Gilman, *Inequalities in discrete subgroups of $PSL(2,R)$,* Canadian Journal of Mathematics **40** (1988), 114–130.

[11] N. A. Isachenko, *Systems of generators of subgroups of $PSL(2,R)$,* Siberian Mathematical Journal **31** (1990), 162–165.

[12] T. Jørgensen, *On discrete groups of Möbius transformations,* American Journal of Mathematics **98** (1976), 739–749.

[13] T. Jørgensen, *A note on subgroup of $SL(2,C)$,* The Quarterly Journal of Mathematics. Oxford **33** (1982), 325–332.

[14] G. J. Martin, *On discrete Möbius groups in all dimensions,* Acta Mathematica **163** (1989), 253–289.

[15] P. Tukia, *Differentiability and rigidity of Möbius groups,* Inventiones Mathematicae **82** (1985), 557–578.

[16] P. Tukia and X. Wang, *Discreteness of subgroups of $SL(2,C)$ containing elliptic elements,* Mathematica Scandinavica **91** (2002), 214–220.

[17] X. Wang, *Dense subgroups of n-dimensional Möbius groups,* Mathematische Zeitschrift **243** (2003), 643–651.

[18] X. Wang and W. Yang, *Discreteness criteria of Möbius groups of high dimensions and convergence theorem of Kleinian groups,* Advances in Mathematics **159** (2001), 68–82.

[19] X. Wang and W. Yang, *Discreteness criteria for subgroups in $SL(2,C)$,* Mathematical Proceedings of the Cambridge Philosophical Society **124** (1998), 51–55.

[20] X. Wang and W. Yang, *Dense subgroups and discrete subgroups in $SL(2,C)$,* The Quarterly Journal of Mathematics. Oxford **50** (1999), 517–521.

[21] X. Wang and W. Yang, *Generating systems of subgroups in $PSL(2,\Gamma_n)$,* Proceedings of the Edinburgh Mathematical Society **45** (2002), 49–58.

[22] S. C. Wang and Q. Zhou, *On the proper conjugation of Kleinian groups,* Geometriae Dedicata **50** (1995), 1–10.

[23] P. Waterman, *Möbius transformations in several dimensions,* Advances in Mathematics **101** (1993), 87–113.

[24] P. Waterman, *Purely elliptic Möbius groups,* in *Holomorphic Functions and Moduli II* (D. Drasin, C. J. Earle, F. W. Gehring, I. Kra and A. Marden, eds.), Springer-Verlag, New York, 1988, pp. 173–178.

[25] N. J. Wielenberg, *Discrete Möbius groups: Fundamental polyhedra and convergence,* American Journal of Mathematics **99** (1977), 861–877.